INTRODUCTION TO MAGNETISM

Autumn Semester 2022

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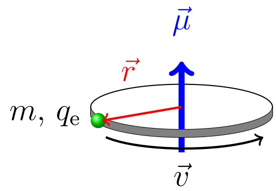
Solution 1

1. Derivation of the elementary magnetic moment in the Ampère description



The electric current i is generally defined as the rate at which charge flows through a given cross-section of a wire, i.e. $i = \frac{dQ}{dt}$. In our system the only charge flowing is $q_{\rm e}$, meaning that the amount of charge $q_{\rm e}$ crosses any surface element perpendicular to the trajectory within the time T. Therefore the current can be written as:

$$i = \frac{q_{\rm e}}{T}\,. \tag{1}$$

The period T can be represented as the circumference of the loop divided by the modulus of the tangential velocity v:

$$T = \frac{2\pi r}{v}\,. \tag{2}$$

Inserting Eq. (2) in Eq. (1) and expressing the area of the loop $\Delta\Sigma$ in terms of its radius $\Delta\Sigma = \pi r^2$, the magnetic moment can be expressed as following:

$$\vec{\mu} = \frac{q_{\rm e} v}{2\pi r}\,\pi r^2\,\hat{n} \;= \frac{q_{\rm e}}{2} v\, r\,\hat{n}\,. \tag{3}$$

Notice that for a planar motion $v\,r\,\hat{n}$ is the same as $\vec{r}\times\vec{v} = \vec{l}/m$, with $\vec{l}$ being the angular momentum of the particle. In conclusion, the magnetic moment $\vec{\mu}$ of a particle with charge $q_{\rm e}$ is related to its angular momentum as following:

$$\vec{\mu} = q_{\rm e}\vec{l}/(2m). \tag{4}$$

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2. Scalar product of two spins 1/2

The generic element $A_{\alpha\beta}$ of the matrix representation $\mathbf{A}$ of the operator $\hat{\mathbf{s}}_1 \cdot \hat{\mathbf{s}}_2$ is given by the "sandwich" of the operator between the *bra* and the *ket* of two generic states $\langle\alpha|$ and $|\beta\rangle$, i.e.

$$A_{\alpha\beta} = \langle\alpha|\hat{\mathbf{s}}_1 \cdot \hat{\mathbf{s}}_2|\beta\rangle\,, \tag{5}$$

where both $\langle\alpha|$ and $|\beta\rangle$ are expressed on the basis $|m_1, m_2\rangle$.
In order to compute such a "sandwich" between two generic states, we need to know how the operator $\hat{\mathbf{s}}_1 \cdot \hat{\mathbf{s}}_2$ acts on the states of our basis $|m_1, m_2\rangle$. Let us write this product in terms of the Pauli matrices:

$$\hat{\mathbf{s}}_1 \cdot \hat{\mathbf{s}}_2 = \frac{1}{4}\hat{\boldsymbol{\sigma}}_1 \cdot \hat{\boldsymbol{\sigma}}_2 = \frac{1}{4}\left(\hat{\sigma}_1^x\hat{\sigma}_2^x + \hat{\sigma}_1^y\hat{\sigma}_2^y + \hat{\sigma}_1^z\hat{\sigma}_2^z\right) \tag{6}$$

where the three terms in parenthesis act to the states of our basis as following:

$\hat{\sigma}_1^x\hat{\sigma}_2^x|++\rangle = |--\rangle$	$\hat{\sigma}_1^y\hat{\sigma}_2^y|++\rangle = +i^2|--\rangle$	$\hat{\sigma}_1^z\hat{\sigma}_2^z|++\rangle = +|++\rangle$
$\hat{\sigma}_1^x\hat{\sigma}_2^x|-+\rangle = |+-\rangle$	$\hat{\sigma}_1^y\hat{\sigma}_2^y|-+\rangle = -i^2|+-\rangle$	$\hat{\sigma}_1^z\hat{\sigma}_2^z|-+\rangle = -|-+\rangle$
$\hat{\sigma}_1^x\hat{\sigma}_2^x|+-\rangle = |-+\rangle$	$\hat{\sigma}_1^y\hat{\sigma}_2^y|+-\rangle = -i^2|-+\rangle$	$\hat{\sigma}_1^z\hat{\sigma}_2^z|+-\rangle = -|+-\rangle$
$\hat{\sigma}_1^x\hat{\sigma}_2^x|--\rangle = |++\rangle$	$\hat{\sigma}_1^y\hat{\sigma}_2^y|--\rangle = +i^2|++\rangle$	$\hat{\sigma}_1^z\hat{\sigma}_2^z|--\rangle = +|--\rangle$

Let us consider the first column. The sandwiches we need to compute are the following:

$$\langle++|\hat{\mathbf{s}}_1 \cdot \hat{\mathbf{s}}_2|++\rangle\,,\langle-+|\hat{\mathbf{s}}_1 \cdot \hat{\mathbf{s}}_2|++\rangle\,,\langle+-|\hat{\mathbf{s}}_1 \cdot \hat{\mathbf{s}}_2|++\rangle\,,\langle--|\hat{\mathbf{s}}_1 \cdot \hat{\mathbf{s}}_2|++\rangle\,. \tag{7}$$

The first element is easily computed without even using the table given for the action of the Pauli matrices. Given:

$$\langle++|\hat{\mathbf{s}}_1 \cdot \hat{\mathbf{s}}_2|++\rangle = \langle++|\frac{1}{4}\left(\hat{\sigma}_1^x\hat{\sigma}_2^x + \hat{\sigma}_1^y\hat{\sigma}_2^y + \hat{\sigma}_1^z\hat{\sigma}_2^z\right)|++\rangle\,, \tag{8}$$

the only non zero contribution comes from the z components $\langle++|\hat{\sigma}_1^z\hat{\sigma}_2^z|++\rangle = 1$.
$\langle++|\hat{\sigma}_1^y\hat{\sigma}_2^y|++\rangle$ and $\langle++|\hat{\sigma}_1^x\hat{\sigma}_2^x|++\rangle$ are zero because the action of $\hat{\sigma}_1^y\hat{\sigma}_2^y$ and $\hat{\sigma}_1^x\hat{\sigma}_2^x$ maps the $|++\rangle$ state into $|--\rangle$ (which is orthogonal to $|++\rangle$ itself).

The **second** and **third** elements in Eq. (7) do not have any contribution from the z components since $\hat{\sigma}_1^z\hat{\sigma}_2^z$ do not change its eigenstate $|++\rangle$. As told before $\hat{\sigma}_1^y\hat{\sigma}_2^y$ and $\hat{\sigma}_1^x\hat{\sigma}_2^x$ map the $|++\rangle$ state into $|--\rangle$, which is orthogonal to both $|-+\rangle$ and $|+-\rangle$. Therefore even these contributions are zeros, and the second and third element of the matrix result to be null.

The fourth in Eq. (7) element has no z contribution as already seen with similar arguments for the second and third, but the contributions $\langle--|\hat{\sigma}_1^y\hat{\sigma}_2^y|++\rangle$ and $\langle--|\hat{\sigma}_1^x\hat{\sigma}_2^x|++\rangle$ are both non-zero. Let us have a look to the matrix at this point, before completing the calculations.

$\hat{\mathbf{s}}_1 \cdot \hat{\mathbf{s}}_2$	$|++\rangle$	$|-+\rangle$	$|+-\rangle$	$|--\rangle$
$\langle++|$	1/4	...	...	...
$\langle-+|$	0	...	...	...
$\langle+-|$	0	...	...	...
$\langle--|$	?	...	...	...

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Now using the same arguments for the first element of the first column, the elements of the matrix which only have a non zero contribution from the z components are the ones in diagonal. Calculating them we have:

$\hat{\mathbf{s}}_1 \cdot \hat{\mathbf{s}}_2$	$\vert{+}{+}\rangle$	$\vert{-}{+}\rangle$	$\vert{+}{-}\rangle$	$\vert{-}{-}\rangle$
$\langle{+}{+}\vert$	$1/4$	$\dots$	$\dots$	$\dots$
$\langle{-}{+}\vert$	0	$-1/4$	$\dots$	$\dots$
$\langle{+}{-}\vert$	0	$\dots$	$-1/4$	$\dots$
$\langle{-}{-}\vert$	?	$\dots$	$\dots$	$1/4$

Using the same arguments for the second and third elements of the first columns:

$\hat{\mathbf{s}}_1 \cdot \hat{\mathbf{s}}_2$	$\vert{+}{+}\rangle$	$\vert{-}{+}\rangle$	$\vert{+}{-}\rangle$	$\vert{-}{-}\rangle$
$\langle{+}{+}\vert$	$1/4$	0	0	$\dots$
$\langle{-}{+}\vert$	0	$-1/4$	$\dots$	0
$\langle{+}{-}\vert$	0	$\dots$	$-1/4$	0
$\langle{-}{-}\vert$	?	0	0	$1/4$

Using the same arguments for the forth element of the first columns:

$\hat{\mathbf{s}}_1 \cdot \hat{\mathbf{s}}_2$	$\vert{+}{+}\rangle$	$\vert{-}{+}\rangle$	$\vert{+}{-}\rangle$	$\vert{-}{-}\rangle$
$\langle{+}{+}\vert$	$1/4$	0	0	?
$\langle{-}{+}\vert$	0	$-1/4$	?	0
$\langle{+}{-}\vert$	0	?	$-1/4$	0
$\langle{-}{-}\vert$	?	0	0	$1/4$

The ? means that the sum of the contributions coming from x and y components needs to be evaluated explicitly. The detailed calculation shows that these contributions are equal and with the same sign for the elements in the center of the matrix and have opposite sign for the elements in the corner resulting in:

$\hat{\mathbf{s}}_1 \cdot \hat{\mathbf{s}}_2$	$\vert{+}{+}\rangle$	$\vert{-}{+}\rangle$	$\vert{+}{-}\rangle$	$\vert{-}{-}\rangle$
$\langle{+}{+}\vert$	$1/4$	0	0	0
$\langle{-}{+}\vert$	0	$-1/4$	$1/2$	0
$\langle{+}{-}\vert$	0	$1/2$	$-1/4$	0
$\langle{-}{-}\vert$	0	0	0	$1/4$

We now change the basis from $|m_1, m_2\rangle$ to $|S, M\rangle$ states and we calculate the elements of the same matrix $\hat{\mathbf{s}}_1 \cdot \hat{\mathbf{s}}_2$ in the new basis. Using the fact that $(\hat{\mathbf{S}})^2 = (\hat{\mathbf{s}}_1 + \hat{\mathbf{s}}_2)^2 = (\hat{\mathbf{s}}_1)^2 + 2\,\hat{\mathbf{s}}_1 \cdot \hat{\mathbf{s}}_2 + (\hat{\mathbf{s}}_2)^2$, i.e:

$$\hat{\mathbf{s}}_1 \cdot \hat{\mathbf{s}}_2 = \frac{(\hat{\mathbf{S}})^2 - (\hat{\mathbf{s}}_1)^2 - (\hat{\mathbf{s}}_2)^2}{2} \tag{9}$$

and knowing that:

$$\begin{aligned} \hat{\mathbf{S}}^2 |S, M\rangle &= S(S+1)|S, M\rangle \\ \hat{\mathbf{s}}_1^2 |S, M\rangle &= s_1(s_1+1)|S, M\rangle \\ \hat{\mathbf{s}}_2^2 |S, M\rangle &= s_2(s_2+1)|S, M\rangle\,, \end{aligned} \tag{10}$$

where $s_1 = s_2 = 1/2$, the matrix results diagonal on the basis $|S, M\rangle$, whose states are thus eigenstates of the operator $\hat{\mathbf{s}}_1 \cdot \hat{\mathbf{s}}_2$:

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$\hat{\mathbf{s}}_1 \cdot \hat{\mathbf{s}}_2$	$\vert 1,+1\rangle$	$\vert 1,0\rangle$	$\vert 1,-1\rangle$	$\vert 0,0\rangle$
$\langle 1,+1\vert$	1/4	0	0	0
$\langle 1,0\vert$	0	1/4	0	0
$\langle 1,-1\vert$	0	0	1/4	0
$\langle 0,0\vert$	0	0	0	−3/4

3. Covariance of the spacetime interval

Given the Lorentz boost along the x direction

$$\begin{cases} x' = \cosh\alpha\, x - \sinh\alpha\,(ct) \\ ct' = \cosh\alpha\,(ct) - \sinh\alpha\, x\,, \end{cases}$$

the difference the two spacetime points $(ct_1, x_1, 0, 0)$ and $(ct_2, x_2, 0, 0)$ is

$$\begin{cases} x_2' - x_1' = \cosh\alpha\,(x_2 - x_1) - \sinh\alpha\, c(t_2 - t_1) \\ c(t_2' - t_1') = \cosh\alpha\,(t_2 - t_1) - \sinh\alpha\,(x_2 - x_1) \end{cases}$$

Thus, taking the square on both sides of the two equations yields

$$\begin{cases} (x_2' - x_1')^2 = [\cosh\alpha\,(x_2 x_1) - \sinh\alpha\, c(t_2 - t_1)]^2 \\ c^2(t_2' - t_1')^2 = [\cosh\alpha\,(t_2 - t_1) - \sinh\alpha\,(x_2 x_1)]^2 \end{cases}$$

so that the equivalence between the squared spacetime intervals can easily be obtained

$$\begin{aligned} c^2(t_2' - t_1')^2 - (x_2' - x_1')^2 &= [\cosh\alpha\, c(t_2 - t_1) - \sinh\alpha\,(x_2 - x_1)]^2 - [\cosh\alpha\,(x_2 - x_1) - \sinh\alpha\, c(t_2 - t_1)]^2 \\ &= \cosh^2\alpha\left[c^2(t_2 - t_1)^2 - (x_2 - x_1)^2\right] - \sinh^2\alpha\left[c^2(t_2 - t_1)^2 - (x_2 - x_1)^2\right] \\ &- 2\cosh\alpha\sinh\alpha\,[c(t_2 - t_1)(x_2 - x_1) - c(t_2 - t_1)(x_2 - x_1)] \\ &= (\cosh^2\alpha - \sinh^2\alpha)\left[c^2(t_2 - t_1)^2 - (x_2 - x_1)^2\right] \\ &= c^2(t_2 - t_1)^2 - (x_2 - x_1)^2 \end{aligned}$$

where the equivalence $\cosh^2\alpha - \sinh^2\alpha = 1$ was used.